Probability Theory

Chapter 5: Continuous Random Variables

Lecturer

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Term 191

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Continuous Random Variables

- So far we have considered discrete random variables that can take on a finite or countably infinite number of values.
- In applications, we are often interested in random variables that can take on an uncountable continuum of values; we call these continuous random variables.
- A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

For Examples:

  - The time until the occurrence of the next phone call at my office;
  - The lifetime of a battery;
  - The height of a randomly selected maple tree;

Continuous Random Variables

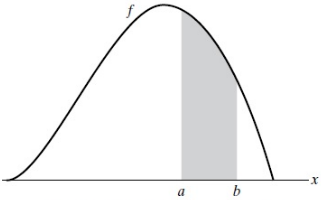
Definition (Continuous Random Variable)

A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

A random variable X is said to be **continuous random variable** if there exists a nonnegative function f , defined for all real $x \in (-\infty, \infty)$, having the property that, for any set B of real numbers,

$$P(X \in B) = \int_B f(x)dx.$$

The function f is called the **probability density function** (pdf) of the random variable X.



$P(a \leq X \leq b)$ = area of shaded region

Figure: Probability density function f, $B = [a, b]$

Continuous Random Variables

- For $B = (-\infty, \infty)$, we have

$$P\left(X \in (-\infty, \infty)\right) = \int_{-\infty}^{\infty} f(x)dx = 1.$$

- For $B = (a, b)$, we have

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx.$$

- For any $a \in \mathbb{R}$, we have

$$P(X = a) = \int_{a}^{a} f(x)dx = 0.$$

- For a continuous random variable,

$$P(X < a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx.$$

Continuous Random Variables

Example

Suppose that X is a continuous random variable whose pdf is given by

$$f(x) = \begin{cases} c(4x - 2x^2) & \text{if } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases},$$

1. What is the value of c?

2. Find $P(X > 1)$.

Continuous Random Variables

Example

A continuous random variable X has the pdf

$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 0.5 \\ \frac{4-2x}{3} & \text{if } 0.5 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}.$$

Find $P(0.25 < X < 1.25)$.

Continuous Random Variables

Example

A continuous random variable X has the pdf

$$f(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} .$$

Find $P(X \leq 2 | X > 1)$.

Continuous Random Variables

Example

The amount of time in hours that a computer functions before breaking down is a continuous random variable with pdf given by

$$f(x) = \begin{cases} \lambda e^{-\frac{x}{100}} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

What is the probability that

1. a computer will function between 50 and 150 hours before breaking down?
2. it will function for fewer than 100 hours?

Continuous Random Variables

Example

The lifetime in hours of a certain kind of radio tube is a random variable having a pdf given by

$$f(x) = \begin{cases} \frac{100}{x^2} & \text{if } x > 100 \\ 0 & \text{otherwise} \end{cases},$$

What is the probability that exactly 2 of 5 such tubes in a radio set will have to be replaced within the first 150 hours of operation? Assume that the events $E_i, i = 1, 2, 3, 4, 5$, that the i^{th} such tube will have to be replaced within this time are independent.

Continuous Random Variables

Definition (Cumulative Distribution Function)

The cumulative distribution function (cdf) of a continuous random variable X is

$$F(x) = P(X \in (-\infty, x]) = P(X \leq x) = \int_{-\infty}^{x} f(u)du, \quad -\infty < x < \infty.$$

The cdf gives the

1. proportion of population with value less than x.

2. probability of having a value less than x.

For example:

If $F(x)$ is the cdf for the age in months of fish in a lake, then $F(10)$ is the probability a random fish is 10 months or younger.

Continuous Random Variables

- Since, $F(x) = \int_{-\infty}^{x} f(u)du$, by "Fundamental Theorem of Calculus" we have

$$\frac{d}{dx}F(x) = f(x).$$

- $P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$.

- $P(X = x) = P(X \leq x) - P(X < x) = 0$.

Continuous Random Variables

Example

Let X be a continuous random variable with pdf given by

$$f(x) = \begin{cases} 3x^2 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases},$$

Find $F(x)$. Graph both $f(x)$ and $F(x)$.

Continuous Random Variables

Example

Suppose that a continuous random variable X has the cumulative distribution function $F(x) = \frac{1}{1+e^{-x}}$ for $-\infty < x < \infty$. Find X's density function.

Continuous Random Variables

Example

If X is continuous with distribution function F_X and pdf f_X, find the pdf of $Y = 2X$.

Expectation and Variance of Continuous Random Variables

Definition (Mean and Variance of a Continuous Random Variable)

- Suppose X is a continuous random variable with pdf $f(x)$. The **mean or expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

- The variance of X, denoted as $V(X)$ or σ^2 is

$$\sigma^2 = E(X-\mu)^2 = E(X^2) - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2.$$

- The standard deviation of X is $\sigma = \sqrt{\sigma^2}$.

Expectation and Variance of Continuous Random Variables

Example

Find $E(X)$ & $Var(X)$ when the density function of X is

$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Expectation and Variance of Continuous Random Variables

Example

Find $E\left(e^X\right)$ when the density function of X is

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Expectation and Variance of Continuous Random Variables

Theorem

If X is a continuous random variable with pdf $f(x)$, then, for any real-valued function g,

$$E\left(g(X)\right) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

Example

Find $E\left(e^X\right)$ when the density function of X is

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Expectation and Variance of Continuous Random Variables

Theorem

If X is a continuous random variable with mean μ and variance σ^2, a and b are constants, then

- $E(aX + b) = a\mu + b$.

- $Var(aX + b) = a^2\sigma^2$.

The Uniform Random Variable

Definition (Uniform Distribution)

A random variable X is said to be uniformly distributed over the interval (α, β), if its *pdf* is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases},$$

Theorem

$f(x)$ is a probability density function.
proof:

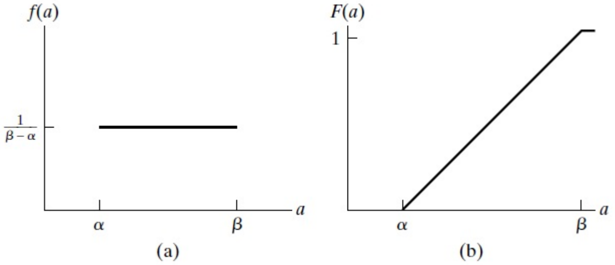
- $X \sim U(\alpha, \beta)$.
- $X \sim U(0, 1)$: standard uniform distribution.

The Uniform Random Variable

Definition (Cumulative Distribution Function)

The *cdf* of the uniform random variable X over the interval (α, β) is given by

$$F(x) = \begin{cases} 0 & x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \alpha \leq x < \beta \\ 1 & x \geq \beta \end{cases},$$



Graph of (a) $f(a)$ and (b) $F(a)$ for a uniform (α, β) random variable.

The Uniform Random Variable

Theorem

Let X be a Uniform random variable with parameter α and β ($X \sim U(\alpha, \beta)$).

1. *The mean of X is given by*

$$\mu = E(X) = \frac{\alpha + \beta}{2},$$

2. *The variance of X is given by*

$$\sigma^2 = Var(X) = \frac{(\beta - \alpha)^2}{12}.$$

Proof:

The Uniform Random Variable

Example

If X is uniformly distributed over $(0, 10)$, calculate the following:

- $P(X < 3)$.
- $P(3 < X < 8)$.
- $E(X)$.
- $Var(X)$.

The Uniform Random Variable

Example

Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

- less than 5 minutes for a bus;

- more than 10 minutes for a bus.

The Uniform Random Variable

Example

Let X be a random variable with a continuous uniform distribution on the interval $(1, a)$ where $a > 1$. If $\mu = 6\sigma^2$, find a.

Normal Random Variables

Definition (Standard Normal Distribution)

A random variable X has the **standard normal distribution** if its *pdf* is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty.$$

Theorem

$f(x)$ is a probability density function.
proof:

- *Need to show $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \sqrt{2\pi}$.*
- *$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy\right) = \int_{\mathbb{R}^2} e^{-\frac{1}{2}(x+y)^2} dxdy$ (By Fubini Theorem).*
- *Pass to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, and $dxdy = rdrd\theta$.*
- *$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{1}{2}r^2} rd\theta dr = 2\pi\int_0^{\infty} re^{-\frac{1}{2}r^2} dr$.*
- *Set $u = \frac{1}{2}r^2$, $du = rdr$, then*
- *$I^2 = -2\pi e^{-\frac{1}{2}r^2}|_0^{\infty} = 2\pi$. Hence, $I = \sqrt{2\pi}$.*

Normal Random Variables

Theorem

For $X \sim N(0,1)$, $E(X) = 0$ and $Var(X) = 1$.

Proof:

1. $E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}x^2} dx = -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} |_{-\infty}^{\infty} = 0.$

2. $Var(X) = E(X^2) - (E(X))^2 = E(X^2).$

   - $E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}x^2} dx$
   - *Set* $u = x$ *and* $dv = x e^{-\frac{1}{2}x^2} dx.$
   - *Then,*

$$Var(X) = \frac{1}{\sqrt{2\pi}} \left(-x e^{-\frac{1}{2}x^2} |_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx \right) \quad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = 1.$$

Normal Random Variables

cdf of $N(0,1)$

The *cdf* of $N(0,1)$ is denoted by $\Phi(x)$.

$$\Phi(x) = P(X \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz.$$

- $\Phi(x)$ Cannot be expressed in terms of elementary functions. It is a special function, tabulated on Page 190 of Ross.
- $\Phi(-x) = 1 - \Phi(x)$ by symmetry.
- Equivalently, $P(X \leq -x) = P(X > x)$.
- Using Mathematical Software (such as Maple), $\Phi(x)$ Can be expressed in terms of error function (another special function) as

$$\Phi(x) = \frac{1}{2}\left(1 + erf\left(\frac{x}{\sqrt{2}}\right)\right).$$

Normal Random Variables

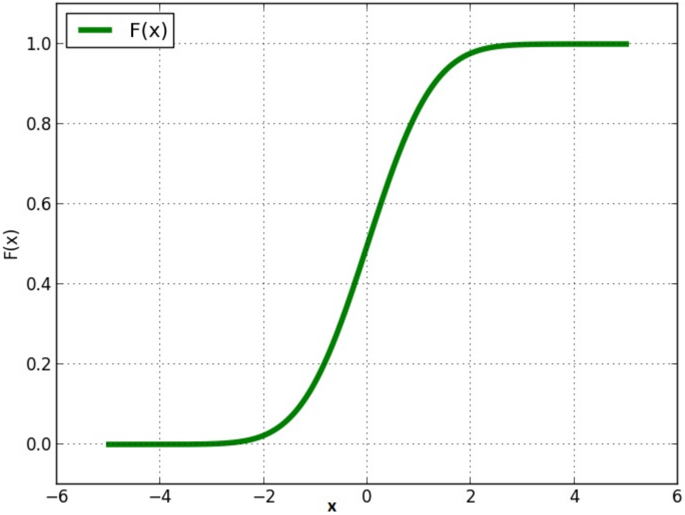


Figure: CDF of Standard Normal RV

Normal Random Variables

Example

Let $X \sim N(0,1)$. Compute $P(|X| < 2)$.

$$\begin{aligned} P(|X| < 2) = P(-2 < X < 2) &= \Phi(2) - \Phi(-2) \\ &= \Phi(2) - (1 - \Phi(2)) \\ &= 2\Phi(2) - 1 \\ &= 0.954. \end{aligned}$$

Example

Let $X \sim N(0,1)$. Compute $P(|X| > 2)$.

$$\begin{aligned} P(|X| > 2) = P(X < -2) + P(X > 2) &= \Phi(-2) + (1 - \Phi(2)) \\ &= 1 - \Phi(2) + 1 - \Phi(2) \\ &= 2\,(1 - \Phi(2)) \\ &= 0.046. \end{aligned}$$

Normal Random Variables

General Normal Distribution

Let $X = \sigma Z + \mu$, where $\mu \in \mathbb{R}$, and $\sigma \in \mathbb{R}^+$. If $Z \sim N(0, 1)$, then $X \sim N(\mu, \sigma^2)$.

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(\sigma Z + \mu \leq x) \\ &= P(Z \leq \frac{x - \mu}{\sigma}) \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

where $\Phi(\cdot)$ is the *cdf* of $N(0, 1)$. By differentiation, the density function of X is then

$$\begin{aligned} f_X(x) &= \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}. \end{aligned}$$

- $E(X) = E(\sigma Z + \mu) = \sigma E(Z) + \mu = \mu$.
- $Var(\sigma Z + \mu) = \sigma^2 Var(Z) = \sigma^2$.

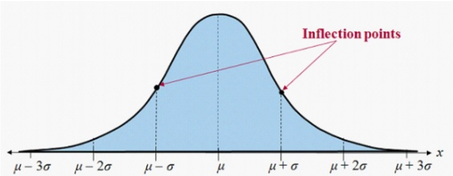
Normal Random Variables

Definition (Normal probability density function)

A random variable X has normal distribution with parameters μ and σ^2 if X has *pdf*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

- μ: Location parameter.
- σ: Scale parameter.
- $X \sim N(\mu, \sigma^2)$



Normal Random Variables

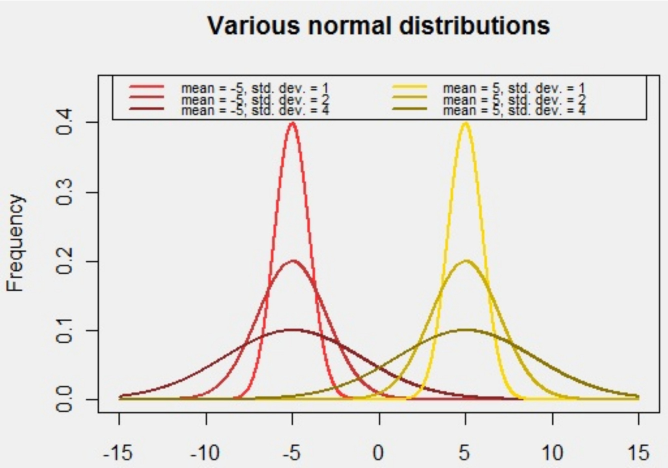


Figure: Meaning of Parameters μ and σ

Normal Random Variables

Definition (Z-Score)

Let X be a random variable with $E(X) = \mu$ and $Var(X) = \sigma^2$. Consider the random variable $Z = \frac{X-\mu}{\sigma}$. Then

$$E(Z) = 0, \quad \text{and} \quad Var(Z) = 1.$$

Z is called the "Z-score" or the standard score of X.

- If $X \sim N(\mu, \sigma^2)$, then $Z \sim N(0, 1)$.
- Advantages of Z:
  1. No units.
  2. Its *pdf* does not depend on any parameters.

Normal Random Variables

Theorem

Suppose $X \sim N(\mu, \sigma^2)$*. Let* $Y = aX + b$*, where* $a, b \in \mathbb{R}$*. Then* $Y \sim N(a\mu + b, a^2\sigma^2)$*.*

Proof:

- *Suppose* $a > 0$ *(The proof when* $a < 0$ *is similar).*
- $F_Y(y) = P(Y \leq y) = F_X\left(\frac{y-b}{a}\right)$.
- *By differentiation, the pdf of* Y *is then*

$$
\begin{aligned}
f_Y(y) &= \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \\
&= \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-b-a\mu}{a\sigma}\right)^2}
\end{aligned}
$$

- *That is,* $Y \sim N(a\mu + b, a^2\sigma^2)$*.*

Normal Random Variables

Example

If $X \sim N(3, 9)$, find

1. $P(2 < X < 5)$.

2. $P(X > 0)$.

3. $P(|X - 3| > 6)$.

Normal Random Variables

Example

The GRE scores are normally distributed with mean 500 and standard deviation 100. What score would place a student in the top 10%.

Normal Random Variables

Example

An examination is frequently regarded as being good if the test scores of those taking the examination can be approximated by a normal density function. The instructor often uses the test scores to estimate the normal parameters μ and σ^2 and then assigns the letter grade *A* to those whose test score is greater than $\mu + \sigma$, *B* to those whose score is between μ and $\mu + \sigma$, *C* to those whose score is between $\mu - \sigma$ and σ, *D* to those whose score is between $\mu - 2\sigma$ and $\mu - \sigma$, and *F* to those getting a score below $\mu - 2\sigma$. (This strategy is sometimes referred to as grading "on the curve.")

(1) $P(X > \mu + \sigma)$.

(2) $P(\mu < X < \mu + \sigma)$.

Normal Random Variables

Example

(3) $P(\mu - \sigma < X < \mu)$.

(4) $P(\mu - 2\sigma < X < \mu - \sigma)$.

(5) $P(X < \mu - 2\sigma)$.

It follows that approximately 16% of the class will receive an *A* grade, 34% a *B* grade, 34% a *C* grade, and 14% a *D* grade; 2% will fail.

The Normal Approximation to the Binomial Distribution

Theorem (The DeMoivre-Laplace limit theorem)

If S_n denotes the number of successes that occur when n independent trials, each resulting in a success with probability p, are performed, then, for any $a < b$,

$$P\left(a \leq \frac{S_n - np}{\sqrt{np(1-p)}}\right) \to \Phi(b) - \Phi(a)$$

as $n \to \infty$.

- It was proved originally for the special case of $p = 0.5$ by DeMoivre in 1733.
- The proof was extended to general p by Laplace in 1812.
- The approximation is good for $np > 5$ and $n(1-p) > 5$ (or equivalently $np(1-p) \geq 10$).

The Normal Approximation to the Binomial Distribution

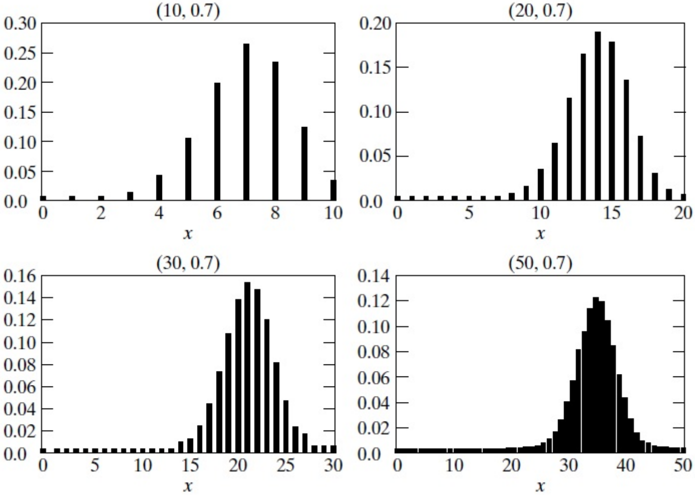


Figure: The probability mass function of a binomial (n, p) random variable becomes more and more "normal" as n becomes larger and larger.

The Normal Approximation to the Binomial Distribution

- To approximate a binomial probability with a normal distribution, a continuity correction is applied as follows:

$$P(X = x) = P(x - 0.5 < X < x + 0.5) \approx P\left(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} < Z < \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

$$P(X \leq x) = P(X \leq x + 0.5) \approx P\left(Z < \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(X \geq x) = P(X \geq x - 0.5) \approx P\left(Z > \frac{x - 0.5 - np}{\sqrt{np(1-p)}}\right)$$

The Normal Approximation to the Binomial Distribution

Example

The manufacturing of semiconductor chips produces 2% defective chips. Assume the chips are independent and that a lot contains 1000 chips.

1. Approximate the probability that more than 25 chips are defective.

2. Approximate the probability that between 20 and 30 chips are defective.

The Normal Approximation to the Binomial Distribution

Example

Suppose that X is a binomial random variable with $n = 200$ and $p = 0.3$.

1. Approximate the probability that X is at most 50.

2. Approximate the probability that $X = 60$ (Also find the exact solution).

Exponential Random Variables

Definition (Exponential Distribution)

Let the random variable X be equal the distance between successive events of a Poisson process with mean number of events $\lambda > 0$ per unit interval, then X is an exponential random variable with parameter λ. The *pdf* of X is given,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

Theorem

$f(x)$ is a probability density function.
proof:

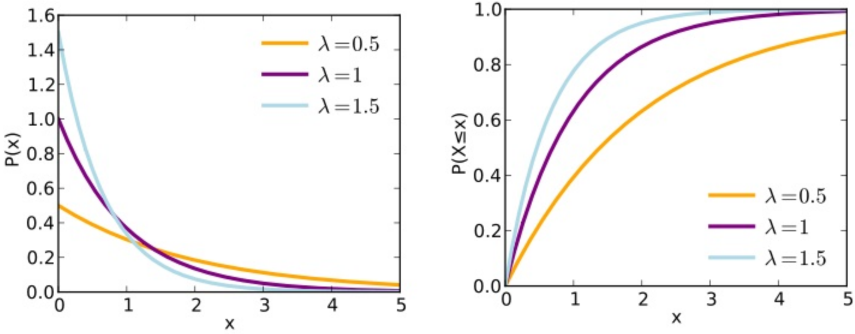
The Exponential Random Variable

Definition (Cumulative Distribution Function)

The cumulative distribution function $F(x)$ of an exponential random variable is given by

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases},$$

The Exponential Random Variable



The Exponential Random Variable

Theorem

Let X be a Exponential random variable with parameter λ, $X \sim Exp(\lambda)$.

1. *The mean of X is given by $\mu = E(X) = \frac{1}{\lambda}$.*
2. *The variance of X is given by $\sigma^2 = Var(X) = \frac{1}{\lambda^2}$.*

Proof: *Find $E(X^n)$.*

Example

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

1. more than 10 minutes;

2. between 10 and 20 minutes.

The Exponential Random Variable

Example

The number of defective parts in the output of a machine is approximately a Poisson process at a mean rate of 30 defectives per hour. What is the probability that we have to wait more than 3 minutes to find the next defective part?

The Exponential Random Variable

Definition (Memoryless Property)

We say that a nonnegative random variable X is memoryless if

$$P(X > s + t | X > t) = P(X > s), \quad \text{for all } s, t \geq 0.$$

- If we think of X as being the lifetime of some instrument, memoryless property states that the probability that the instrument survives for at least $s + t$ hours, given that it has survived t hours, is the same as the initial probability that it survives for at least s hours.
- In other words, if the instrument is alive at age t, the distribution of the remaining amount of time that it survives is the same as the original lifetime distribution.
- Equivalent relations:

$$\frac{P(X > s + t, X > t)}{P(X > t)} = P(X > s) \to P(X > s + t) = P(X > t)P(X > s).$$

- Exponentially distributed random variables are memoryless in the sense that

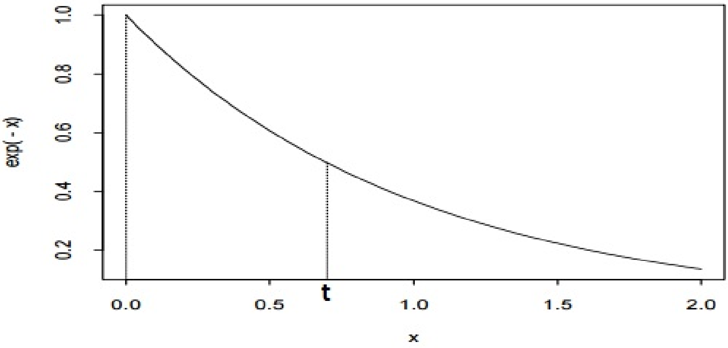
$$e^{-\lambda(s+t)} = e^{-\lambda t} e^{-\lambda s}.$$

The Exponential Random Variable

Memoryless Property

- The graph after the point t is an exact copy of the original function.

- The important consequence of this is that the distribution of X conditioned on $\{X > t\}$ is again exponential.



The Exponential Random Variable

Example

Suppose that the amount of time one spends in a bank is exponentially distributed with mean 10 minutes.

1. What is the probability that a customer will spend more than 15 minutes in the bank?

2. What is the probability that a customer will spend more than 15 minutes in the bank given that he is still in the bank after 10 minutes?

The Exponential Random Variable

Definition (Laplace Distribution)

A variation of the exponential distribution is the distribution of a random variable that is equally likely to be either positive or negative and whose absolute value is exponentially distributed with parameter λ, $\lambda \geq 0$. Such a random variable is said to have a **Laplace distribution**, and its density is given by

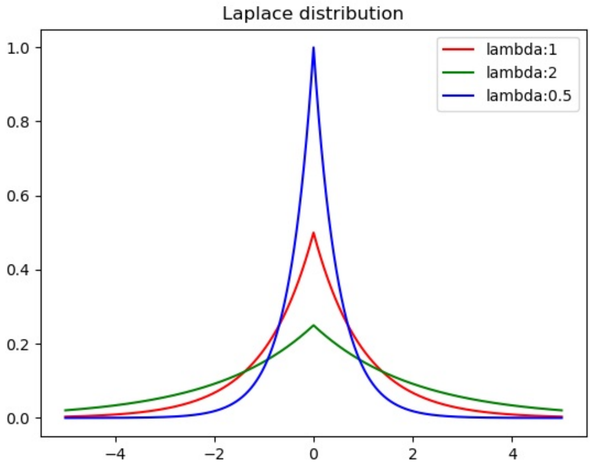
$$f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}, \quad -\infty < x < \infty.$$

- Its distribution function is given by

$$F(x) = \begin{cases} \frac{1}{2}e^{\lambda x} & x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & x \geq 0 \end{cases},$$

- Sometimes it is called the **double exponential distribution**.

The Exponential Random Variable



The Gamma Distribution

- The gamma distribution can be viewed as a generalization of the exponential distribution with mean $\frac{1}{\lambda}$, $\lambda > 0$.
- An exponential random variable with mean $\frac{1}{\lambda}$ represents the waiting time until the 1^{st} event to occur, where events are generated by a Poisson process with mean λ.
- While the gamma random variable X represents the waiting time until the α^{th} event to occur.
- Therefore, $X = \sum_{i=1}^{\alpha} Y_i$, where $Y_1, \ldots, Y_n$ are independent exponential random variables with mean $\frac{1}{\lambda}$.

The Gamma Distribution

- The probability density function of X is given by:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)}\lambda e^{-\lambda x}(\lambda x)^{\alpha-1} & x > 0 \\ 0 & x \geq 0 \end{cases},$$

- $X \sim Ga(\alpha, \lambda)$.

  1. α is the shape parameter.
  2. λ scale parameter.

The Gamma Distribution

- $\Gamma(\alpha)$ is called the gamma function, is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy.$$

- Integration of $\Gamma(\alpha)$ by parts yields

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1).$$

- For integer values of α, say $\alpha = n$, we obtain

$$\Gamma(n) = (n - 1)!.$$

- $\Gamma(1) = 1$.

The Gamma Distribution

- Let T_n denote the time at which the n^{th} event occurs.
- Our goal is to know what is the distribution of T_n. That is, $F(t) = P(T_n \le t)$.
- Note that $T_n \le t$ if and only if the number of events that have occurred by time t is at least n.
- That is, with $N(t)$ equal to the number of events in $[0, t]$,

$$\begin{aligned} F(t) = P(T_n \le t) &= P(N(t) \ge n) \\ &= \sum_{j=n}^{\infty} P(N(t) = j) \\ &= \sum_{j=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \end{aligned}$$

The Gamma Distribution

- Differentiation yields the density function of T_n (**HW**):

$$f(t) = \frac{\lambda e^{\lambda t} (\lambda t)^{n-1}}{(n-1)!}.$$

- This distribution is often referred to in the literature as the *n*-Erlang distribution.

- Note that when $n = 1$, this distribution reduces to the exponential distribution.

- The gamma distribution with $\lambda = \frac{1}{2}$ and $\alpha = \frac{n}{2}$, n a positive integer, is called the χ_n^2 (read "chi-squared") distribution with n **degrees of freedom**.

The Gamma Distribution

Theorem

Let X be a gamma random variable with parameters α and λ, $X \sim Ga(\alpha, \lambda)$.

1. *The mean of X is given by $\mu = E(X) = \frac{\alpha}{\lambda}$.*
2. *The variance of X is given by $\sigma^2 = Var(X) = \frac{\alpha}{\lambda^2}$.*

Proof:

The Cauchy Distribution

- The pdf of a Cauchy distribution with the **location parameter** a, $-\infty < a < \infty$, and the **scale parameter** b, $b > 0$, is given by

$$f(x) = \frac{1}{\pi b \left[1 + \left(\frac{x-a}{b}\right)^2\right]}, \quad -\infty < x < \infty.$$

- The cumulative distribution function can be expressed as

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x-a}{b}\right), \quad -\infty < x < \infty.$$

- The standard Cauchy distribution function can be obtained by replacing a with 0 and b with 1.

- Mean and the moments in general do not exist.

The Beta Distribution

- A random variable is said to have a beta distribution if its density is given by

$$f(x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1.$$

Theorem

Let X be a beta random variable with parameters a and b, $X \sim Be(a, b)$.

1. *The mean of X is given by $\mu = E(X) = \frac{a}{a+b}$.*
2. *The variance of X is given by $\sigma^2 = Var(X) = \frac{ab}{(a+b)^2(a+b+1)}$.*

Proof: (HW)

- Suppose the distribution of X is $f_X(x)$.
- Let $Y = g(X)$.
- Our goal in this section is to find $f_Y(y)$.
- In this section we discuss the **Distribution Function Technique**.
- We illustrate with the following examples.

The Distribution of a Function of a Random Variable

Example

Let X be uniformly distributed over (0, 1). Let $Y = X^n$, find $f_Y(y)$.
Solution:

- $F_Y(y) = y^{\frac{1}{n}}$.
- $f_Y(y) = \frac{1}{n} y^{\frac{1}{n}-1}, \quad 0 < y < 1.$

The Distribution of a Function of a Random Variable

Example

If X is a continuous random variable with probability density f_X, Let $Y = X^2$, find $f_Y(y)$.
Solution:

- $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$.
- $f_Y(y) = \frac{1}{2\sqrt{y}} \left[f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right], \quad y \geq 0$.

The Distribution of a Function of a Random Variable

Theorem

Let X be a continuous random variable having pdf f_X. Suppose that $g(x)$ is a strictly monotonic (increasing or decreasing), differentiable (and thus continuous) function of x. Then the random variable Y defined by $Y = g(X)$ has a pdf given by

$$f_Y(y) = \left|\frac{d}{dy}g^{-1}(y)\right| f_X\left(g^{-1}(y)\right),$$

where $g^{-1}(y)$ is defined to equal that value of x such that $g(x) = y$.

Proof: (HW)

Problems and Exercises

PROBLEMS

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THEORETICAL EXERCISES

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5, 10, 11, 12, 13, 14, 15, 18, 19, 21, 26, 27, 31, 33