Gamma Distribution

- The exp r.v. represents the waiting time until the 1^{st} event occur.
- In the gamma dist, the r.v Y represents the waiting time until "r" events occur.
- We want to find the pdf of Y.
- $F_Y(y) = P(Y \le y) = 1 - P(Y > y)$

The event $\{Y > y\}$ means the waiting time until the "r" events occur is greater than y.
This event occurs iff there are $(r-1)$ or less events in the interval $[0, y]$

$\le r-1$ events

$0 \longrightarrow time \longrightarrow y$

$$= 1 - P(X \le r-1) \quad \text{where } X \sim P_o(\underbrace{\lambda y}_{\text{the rate in } [0,y]})$$

$$= 1 - \sum_{x=0}^{r-1} \frac{e^{-\lambda y} (\lambda y)^x}{x!}$$

- Differentiate w.r.t. y we get:

$$f_Y(y) = 0 - \sum_{x=0}^{r-1} \left\{ \frac{1}{x!} \frac{d}{dy} \left[e^{-\lambda y} (\lambda y)^x \right] \right\}$$

$$= - \sum_{x=0}^{r-1} \frac{1}{x!} \left[e^{-\lambda y} \cdot x (\lambda y)^{x-1} \lambda + (\lambda y)^x e^{-\lambda y} \cdot (-\lambda) \right]$$

$$= \sum_{x=0}^{r-1} \left\{ \frac{1}{x!} \cdot \lambda e^{-\lambda y} (\lambda y)^{x-1} \left[\lambda y - x \right] \right\}$$

$$= \lambda e^{-\lambda y} \sum_{x=0}^{r-1} \frac{(\lambda y)^{x-1}}{x!} \left[\lambda y - x \right]$$

$$= \lambda e^{-\lambda y} \left[\frac{(\lambda y)^{-1}}{0!} \cdot [\lambda y - 0] + \sum_{x=1}^{r-1} \frac{(\lambda y)^{x-1}}{x!} [\lambda y - x] \right]$$

$$= \lambda e^{-\lambda y} + \lambda e^{-\lambda y} \sum_{x=1}^{r-1} \frac{(\lambda y)^{x-1}}{x!} [\lambda y - x]$$

$$= \lambda e^{-\lambda y} + \lambda e^{-\lambda y}\left[(\lambda y - 1) + \frac{\lambda y}{2}(\lambda y - 2) + \frac{(\lambda y)^2}{3!}(\lambda y - 3) + \cdots + \frac{(\lambda y)^{r-2}}{(r-1)!}(\lambda y - (r-1)) \right]$$

$$= \lambda e^{-\lambda y} + \lambda e^{-\lambda y}\left[\lambda y - 1 + \frac{(\lambda y)^2}{2} - \lambda y + \frac{(\lambda y)^3}{3!} - \frac{(\lambda y)^2}{2} + \cdots + \frac{(\lambda y)^{r-1}}{(r-1)!} - \frac{(\lambda y)^{r-2}}{(r-2)!} \right]$$

$$= \lambda e^{-\lambda y} \cdot \frac{(\lambda y)^{r-1}}{(r-1)!} = \frac{\lambda^r y^{r-1} e^{-\lambda y}}{(r-1)!}$$

$$= \frac{\lambda^r y^{r-1} e^{-\lambda y}}{\Gamma(r)}.$$

- When $r = 1$, the exp. dist. is obtained.
- When $r = \frac{n}{2}$ & $\lambda = \frac{1}{2}$, the χ^2 dist. with n degrees of freedom is obtained.

  ↳ Chi-squared

- Let $X \sim Ga(\alpha, \lambda)$, then:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx = \int_0^{\infty} x \cdot \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}\, dx$$

$$= \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} \underbrace{x^\alpha \cdot \lambda^\alpha}_{(x\lambda)^\alpha} e^{-\lambda x}\, dx$$

Let $y = \lambda x$

$dy = \lambda\, dx$

$$= \frac{1}{\lambda \Gamma(\alpha)} \int_0^{\infty} y^\alpha e^{-y}\, dy$$

$$= \frac{1}{\lambda \Gamma(\alpha)} \cdot \Gamma(\alpha+1) = \frac{\alpha \Gamma(\alpha)}{\lambda \Gamma(\alpha)} = \frac{\alpha}{\lambda} \#.$$

- For χ^2-dist: $E(X) = n$; $var(X) = 2n$.