

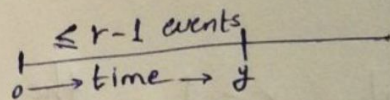
Gamma Distribution

- In the exp. r.v. X represents the waiting time until the 1st event occur.
- In the gamma dist., the r.v. Y represents the waiting time until "r" events occur.
- We want to find the pdf of Y .

$$F_Y(y) = P(Y \leq y) = 1 - P(Y > y)$$

The event $\{Y > y\}$ means the waiting time until the "r" events occur is greater than y .

This event occurs iff there are $(r-1)$ or less events in the interval $[0, y]$



$$= 1 - P(X \leq r-1) \quad \text{where } X \sim P_0(\lambda y)$$

λ → the rate in $[0, y]$

$$= 1 - \sum_{x=0}^{r-1} \frac{e^{-\lambda y} (\lambda y)^x}{x!}$$

- Differentiate w.r.t. y we get:

$$f_Y(y) = 0 - \sum_{x=0}^{r-1} \left\{ \frac{1}{x!} \frac{d}{dy} [e^{-\lambda y} (\lambda y)^x] \right\}$$

$$= - \sum_{x=0}^{r-1} \frac{1}{x!} \left[e^{-\lambda y} \cdot x (\lambda y)^{x-1} \lambda + (\lambda y)^x e^{-\lambda y} \cdot (-\lambda) \right]$$

$$= \sum_{x=0}^{r-1} \left\{ \frac{1}{x!} \cdot \lambda e^{-\lambda y} (\lambda y)^{x-1} [\lambda y - x] \right\}$$

$$= \lambda e^{-\lambda y} \sum_{x=0}^{r-1} \frac{(\lambda y)^{x-1}}{x!} [\lambda y - x]$$

$$= \lambda e^{-\lambda y} \left[\frac{(\lambda y)^{-1}}{0!} \cdot [\lambda y - 0] + \sum_{x=1}^{r-1} \frac{(\lambda y)^{x-1}}{x!} [\lambda y - x] \right]$$

$$= \lambda e^{-\lambda y} + \lambda e^{-\lambda y} \sum_{x=1}^{r-1} \frac{(\lambda y)^{x-1}}{x!} [\lambda y - x]$$

$$\begin{aligned}
&= \lambda e^{-\lambda y} + \lambda e^{-\lambda y} \left[(\lambda y - 1) + \frac{\lambda y}{2} (\lambda y - 2) + \right. \\
&\quad \left. \frac{(\lambda y)^2}{3!} (\lambda y - 3) + \dots + \frac{(\lambda y)^{r-2}}{(r-1)!} (\lambda y - (r-1)) \right] \\
&= \lambda e^{-\lambda y} + \lambda e^{-\lambda y} \left[\lambda y - 1 + \frac{(\lambda y)^2}{2} - \lambda y \right. \\
&\quad \left. + \frac{(\lambda y)^3}{3!} - \frac{(\lambda y)^2}{2} + \dots + \frac{(\lambda y)^{r-1}}{(r-1)!} - \frac{(\lambda y)^{r-2}}{(r-2)!} \right] \\
&= \lambda e^{-\lambda y} \cdot \frac{(\lambda y)^{r-1}}{(r-1)!} = \frac{\lambda^r y^{r-1} e^{-\lambda y}}{(r-1)!} \\
&= \frac{\lambda^r y^{r-1} e^{-\lambda y}}{\Gamma(r)}.
\end{aligned}$$

- When $r=1$, the exp. dist. is obtained.

- When $r = \frac{n}{2}$ & $\lambda = \frac{1}{2}$, the χ^2 dist. with n degrees of freedom is obtained.
↳ Chi-squared

- Let $X \sim \text{Ga}(\alpha, \lambda)$, then:

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_0^{\infty} x \cdot \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx \\
&= \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} \underbrace{x^\alpha \cdot \lambda^\alpha}_{(x\lambda)^\alpha} e^{-\lambda x} dx \\
&= \frac{1}{\lambda \Gamma(\alpha)} \int_0^{\infty} y^\alpha e^{-y} dy \quad \begin{array}{l} \text{Let } y = \lambda x \\ dy = \lambda dx \end{array} \\
&= \frac{1}{\lambda \Gamma(\alpha)} \cdot \Gamma(\alpha+1) = \frac{\alpha \Gamma(\alpha)}{\lambda \Gamma(\alpha)} = \frac{\alpha}{\lambda} \quad \#
\end{aligned}$$

- For χ^2 dist.: $E(X) = n$; $\text{var}(X) = 2n$.