Probability and Statistics for Engineers

Chapter 4: Continuous Random Variables and Probability Distributions

Lecturer

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Term 192

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Continuous Random Variables

- So far we have considered discrete random variables that can take on a finite or countably infinite number of values.

- In applications, we are often interested in random variables that can take on an uncountable continuum of values; we call these continuous random variables.

Definition (Continuous Random Variable)

A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

For Examples:

- The time until the occurrence of the next phone call at my office;

- The lifetime of a battery;

- The height of a randomly selected maple tree;

Probability Density Functions

Definition

For a continuous random variable X, a **probability density function (pdf)** is a function such that

1. $f(x) \geq 0$.
2. $\int_{-\infty}^{\infty} f(x) = 1$.
3. $P(a < X < b) = \int_a^b f(x)$.

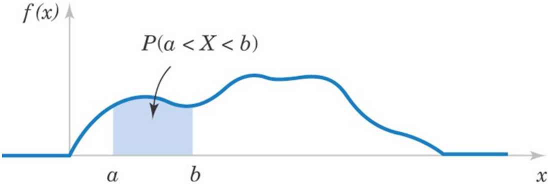


Figure: Probability determined from the area f(x)

Probability Density Functions

- For a continuous random variable X and any value x,

$$p(X = x) = 0.$$

- If X is a continuous random variable, for any x_1 and x_2,

$$P(x_1 \leq X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X \leq x_2) = P(x_1 < X < x_2)$$

Probability Density Functions

Example

Suppose that $f(x) = \frac{c}{256}(8x - x^2)$ for $0 < x < 8$. Determine the constant c.

Probability Density Functions

Example

For the previous example, determine the following:

1. $P(X < 2)$.

2. $P(X > 6)$.

3. $P(2 < X < 6)$.

Probability Density Functions

Example

For the previous example, determine a such that $P(X < a) = 0.95$.

Probability Density Functions

Example

The probability density function of the time to failure of an electronic component in a copier (in hours) is

$$f(x) = \frac{1}{1000} e^{\frac{-x}{1000}}, \quad x > 0$$

Determine the probability that

1. A component lasts more than 3000 hours before failure.
2. A component fails in the interval from 1000 to 2000 hours.

Probability Density Functions

Example

In the previous example, determine the number of hours at which 10% of all components have failed.

Cumulative Distribution Functions

Definition

The cumulative distribution function (cdf) of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du, \quad -\infty < x < \infty.$$

The cdf gives the

1. proportion of population with value less than x.
2. probability of having a value less than x.

For example:

If $F(x)$ is the cdf for the age in months of fish in a lake, then $F(10)$ is the probability a random fish is 10 months or younger.

Cumulative Distribution Functions

Properties of $F(x)$:

1. $F(x)$ goes to 0 as x gets smaller:

$$\lim_{x \to -\infty} F(x) = 0.$$

2. Conversely:

$$\lim_{x \to \infty} F(x) = 1.$$

3. $F(x)$ is non-decreasing.
   - The derivative is a probability density function, which cannot be negative.
   - Also, $F(4)$ can't be less than $F(3)$, for example.

Cumulative Distribution Functions

Example

Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < -2 \\ 0.25x + 0.5 & \text{if } -2 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases},$$

Determine the following

1. $P(X < 1.8)$.
2. $P(X > -1.5)$.
3. $P(-1 < X < 1)$.
4. the pdf of x.

Cumulative Distribution Functions

Cumulative Distribution Functions

Example

Life expectancy (in days) of electronic component has probability density function,

$$f(x) = \frac{1}{x^2}, \quad x \geq 1.$$

1. Find the cdf for the life expectancy.

Cumulative Distribution Functions

Example

The cumulative distribution function of the random variable X, the time (in days) from the diagnosis age until death for one population of Covid-19 patients, is as follows:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-0.03x^{1.2}} & \text{if } x \geq 0 \end{cases},$$

1. Find the probability that a randomly selected person from this population survives at least 12 days.

2. Find the median of X.

Mean and Variance of a Continuous Random Variable

Definition (Mean and Variance of a Continuous Random Variable)

- Suppose X is a continuous random variable with probability density function $f(x)$. The **mean or expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

- The variance of X, denoted as $V(X)$ or σ^2 is

$$\sigma^2 = E(X-\mu)^2 = E(X^2) - \mu^2 = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2.$$

- The standard deviation of X is $\sigma = \sqrt{\sigma^2}$.

Mean and Variance of a Continuous Random Variable

Example

Suppose $f(x) = 1.5x^2$, $-1 < x < 1$. Determine the

1. mean.
2. variance.
3. standard deviation.

Mean and Variance of a Continuous Random Variable

Example

The probability density function of the weight of packages delivered by a post office is $f(x) = \frac{70}{69x^2}$, $1 < x < 70$ pounds.

1. Determine the mean and variance of weight. **Ans. 4.3101, 16.423**.

2. Determine the probability that the weight of a package exceeds 50 pounds. **Ans. 0.0058**.

Continuous Uniform Distribution

Definition (Continuous Uniform Distribution)

A continuous random variable X with probability density function

$$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b,$$

is a **continuous uniform random variable**.

Definition (Mean and Variance)

If X is a continuous uniform random variable over $a \leq x \leq b$, then

$$\mu = E(X) = \frac{a+b}{2}.$$

$$\sigma^2 = V(X) = \frac{(b-a)^2}{12}.$$

Continuous Uniform Distribution

The **cumulative distribution function** of a continuous uniform random variable is obtained by integration.

$$F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x < b \\ 1 & \text{if } x \geq b \end{cases},$$

Proof:

Continuous Uniform Distribution

Example

Suppose X has a continuous uniform distribution over the interval $[-1, 1]$.

1. Determine the mean, variance, and standard deviation of X. **Ans. 0, 1/3, 0.577**.
2. What is $P(X < 0)$. **Ans. 0.5**.
3. Determine the value for z such that $P(-z < X < z) = 0.90$. **Ans. 0.90**.
4. Determine the cumulative distribution function.

Continuous Uniform Distribution

Continuous Uniform Distribution

Example

Suppose the time it takes a data collection operator to fill out an electronic form for a database is uniformly between 1.5 and 2.2 minutes.

1. What is the mean and variance of the time it takes an operator to fill out the form? **Ans. 1.85 min, 0.0408 min**^{2}.

2. What is the probability that it will take less than two minutes to fill out the form? **Ans. 0.7143**.

3. Determine the cumulative distribution function of the time it takes to fill out the form.

Continuous Uniform Distribution

Normal Distribution

- Most widely used distribution of a random variable.
- Two parameters completely define a normal probability density function, μ and σ^2.
- The probability density function is given by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\}, \quad -\infty < x < \infty.$$

- μ is the expected value (mean), or center of the distribution ($-\infty < \mu < \infty$).
- σ^2 is the variance of the distribution ($\sigma^2 > 0$).
- Normal distribution is also referred to as a **Gaussian distribution**.
- Notation: $X \sim N(\mu, \sigma^2)$.

Normal Distribution

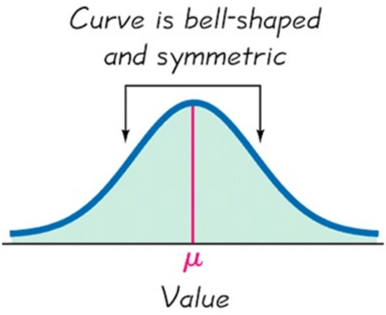


Figure: Normal probability density is symmetric about the mean μ

Normal Distribution

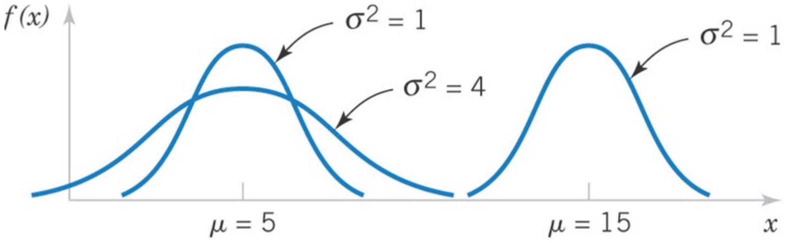


Figure: Normal probability density functions for selected values of the parameters μ and σ^2

Normal Distribution

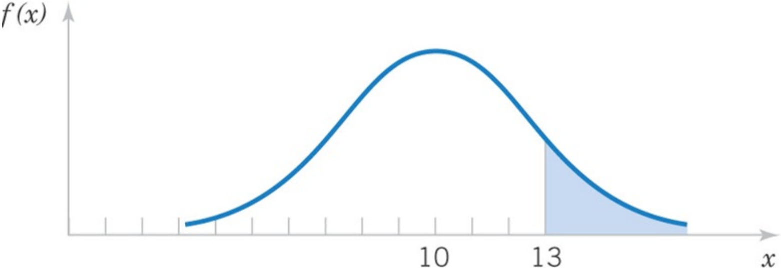


Figure: Probability that X>13 for a normal random variable with $\mu = 10$ and $\sigma^2 = 4$

Normal Distribution

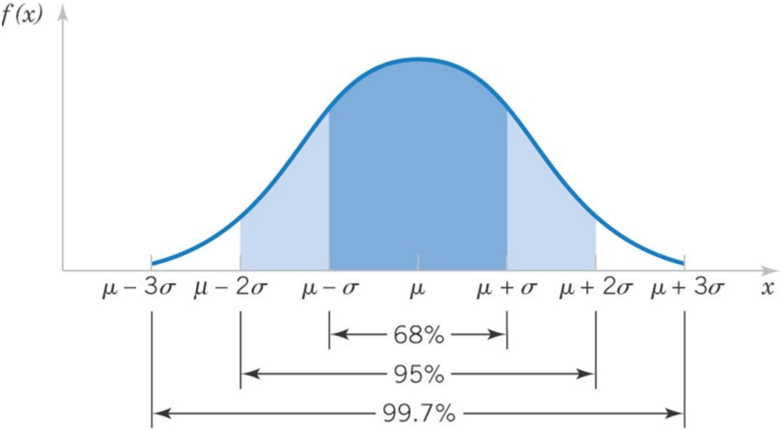


Figure: Probabilities associated with a normal distribution

Normal Distribution

Definition (Standard Normal Distribution)

- A normal random variable with $\mu = 0$ and $\sigma = 1$ is called a **standard normal random variable** and is denoted as Z.

- The cumulative distribution function of a standard normal random variable is denoted as
$$\Phi(z) = P(Z \leq z).$$

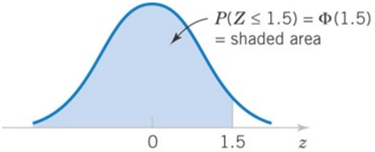
- If $X \sim N(\mu, \sigma^2)$, then $Z \sim N(0, 1)$, where $Z = \frac{X-\mu}{\sigma}$.

- The *pdf* of Z is given by
$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} z^2\}.$$

Normal Distribution

- Appendix Table III (Page 709) provides probabilities of the form $\Phi(z) = P(Z \leq z)$.

- The use of Table III to find $\Phi(1.5) = P(Z \leq 1.5)$ is illustrated in the following figure.



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Figure: Standard normal probability density function

Normal Distribution

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} \, du$$

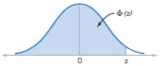


Table III Cumulative Standard Normal Distribution (*continued*)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.532922	0.527903	0.531881	0.535856
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903
0.7	0.758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913
1.0	0.841345	0.843752	0.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888
1.5	0.933193	0.934478	0.935744	0.936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705
2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967

Normal Distribution

Example

Assume Z has a standard normal distribution. Determine the following.

1. $P(Z \leq 1.32)$. **Ans. 0.90658**.
2. $P(Z > 1.45)$. **Ans. 0.07353**.
3. $P(Z > -2.15)$. **Ans. 0.98422**.
4. $P(-2.34 < Z < 1.76)$. **Ans. 0.95116**.
5. Determine z_0 such that $P(Z < z_0) = 0.9$. **Ans. 1.28**.
6. Determine z_0 such that $P(Z > z_0) = 0.1$. **Ans. 1.28**.
7. Determine z_0 such that $P(-1.24 < Z < z_0) = 0.8$. **Ans. 1.33**.
8. Determine z_0 such that $P(-z_0 < Z < z_0) = 0.9973$. **Ans. 3.00**.

Normal Distribution

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z)$$

where Z is a standard normal random variable, and $z = \frac{x-\mu}{\sigma}$ is the z-value obtained by standardizing X.

Normal Distribution

Example

Assume X is normally distributed with a mean of 10 and a standard deviation of 2. Determine the following:

1. $P(X < 13)$. **Ans. 0.841345**.

2. $P(2 < X < 4)$. **Ans. 0.00132**.

3. Determine the value for x such that $P(X > x) = 0.5$. **Ans. 10**.

4. Determine the value for x such that $P(-x < X - 10 < x) = 0.99$. **Ans. 5.16**.

Normal Distribution

Example

The time until recharge for a battery in a laptop computer under common conditions is normally distributed with a mean of 260 minutes and a standard deviation of 50 minutes.

1. What is the probability that a battery lasts more than four hours? **Ans. 0.6554**.

2. What are the quartiles (the 25% and 75% values) of battery life? **Ans. 226.2755, 293.7245**.

Exponential Distribution

Definition (Exponential Distribution)

- The random variable X that equals the distance between successive events of a Poisson process with mean number of events $\lambda > 0$ per unit interval is an **exponential random variable** with parameter λ. The probability density function of X is

$$f(x) = \lambda \exp\{-\lambda x\}, \quad x \geq 0.$$

- The mean of X is

$$E(X) = \frac{1}{\lambda}.$$

- The variance of X is

$$V(X) = \frac{1}{\lambda^2}.$$

Exponential Distribution

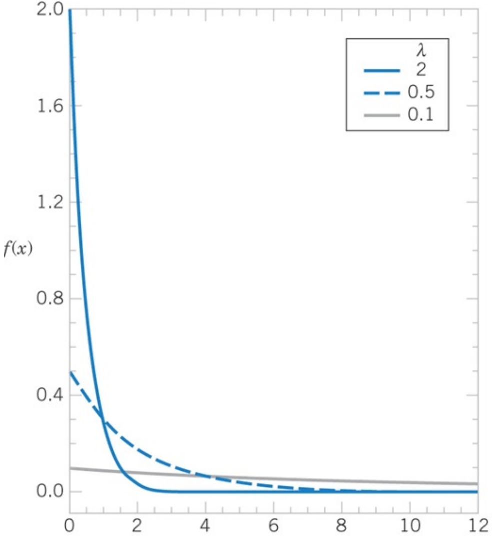


Figure: Probability density function of exponential random variables for selected values of λ

Exponential Distribution

1. **Lack of memory property**: For an exponential random variable X,

$$P(X < t_1 + t_2 | X > t_1) = P(X < t_2).$$

2. **Poisson versus Exponential**: The two distributions are distinct, but both relate to the same process.

3. Given a POISSON PROCESS:

    1. the number of events in a given time period has a POISSON DISTRIBUTION.

    2. the following have an EXPONENTIAL DISTRIBUTION:

        1. The time until the first event.

        2. The time from now until the next occurrence of an event.

        3. The time interval between two successive events.

Example

The time between arrivals of taxis at a busy intersection is exponentially distributed with a mean of 10 minutes.

1. What is the probability that you wait longer than one hour for a taxi? **Ans. 0.0025**.
2. Suppose you have already been waiting for one hour for a taxi. What is the probability that one arrives within the next 10 minutes? **Ans. 0.6321**.
3. Determine x such that the probability that you wait less than x minutes is 0.90. **Ans. 18.97**.

Exponential Distribution

Example

Suppose that the log-ons to a computer network follow a Poisson process with an average of three counts per minute.

1. What is the mean time between counts? **Ans.** $1/3$.

2. Determine the time x such that the probability of at least one count occurs before time x minutes is 0.95. **Ans.** $-3\ln(0.05) = 8.987197$.

3. Determine the length of an interval of time such that the probability of at least one count occurs in the interval is 0.95. **Ans. 8.987197**.